

A GAP FOR EIGENVALUES OF A CLAMPED PLATE PROBLEM

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ABSTRACT. This paper studies eigenvalues of the clamped plate problem on a bounded domain in an n -dimensional Euclidean space. We give an estimate for the gap between $\sqrt{\Gamma_{k+1} - \Gamma_1}$ and $\sqrt{\Gamma_k - \Gamma_1}$, for any positive integer k . According to the asymptotic formula of Agmon and Pleijel, we know, the gap between $\sqrt{\Gamma_{k+1} - \Gamma_1}$ and $\sqrt{\Gamma_k - \Gamma_1}$ is bounded by a term with a lower order $k^{\frac{1}{n}}$ in the sense of the asymptotic formula of Agmon and Peijel, where Γ_j denotes the j^{th} eigenvalue of the clamped plate problem.

1. INTRODUCTION

It is well-known that study on eigenvalues of the eigenvalue problem of elliptic operators is a very important subject in geometry and analysis.

Let Ω be a bounded domain with piecewise smooth boundary in an n -dimensional complete Riemannian manifold M . The following is called *the Dirichlet eigenvalue problem of Laplacian*:

$$(1.1) \quad \begin{cases} \Delta u = -\lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ is the Laplacian on M . Many mathematicians study universal estimates of eigenvalues of the Dirichlet eigenvalue problem of Laplacian. As main developments for study on universal estimates of eigenvalues, Payne, Pólya and Weinberger [24], Hile and Protter [20], Yang [29] makes very important contributions for bounded domains in Euclidean spaces (see Ashbaugh [2, 3, 4]). For domains in sphere, Cheng and Yang [14] obtains optimal universal estimates on eigenvalues. For bounded domains in complete Riemannian manifolds, universal estimates on eigenvalues have been obtained by in Cheng and Yang [15], Chen and Cheng [5], Chen, Zheng and Yang [6] and El Soufi, Harrell and Ilias [19], Cheng [8] and so on. By making use of the universal estimates on eigenvalues and the recursive inequality of Cheng and Yang [17], Cheng and Yang [18] obtain sharp lower bounds and upper bounds for the k^{th} eigenvalues of the Dirichlet eigenvalue problem of Laplacian in the sense of order of k . For bounded domains in the Euclidean space, by making use of Fourier transform, Li and Yau [23] gives an optimal lower bound for the average of the

Key words and phrases: the Dirichlet eigenvalue problem of Laplacian, eigenvalues, eigenfunctions, the clamped plate problem

2010 *Mathematics Subject Classification*: 35P15, 58C40, 53C42.

The first author is supported by NSFC. The second author is partially supported by JSPS Grant-in-Aid for Scientific Research (B): No.16H03937. The third author is supported by NSFC No. 11371150.

first k eigenvalues of the Dirichlet eigenvalue problem of Laplacian. Recently, in [4], Ashbaugh gives a very nice survey for estimates on eigenvalues of the Dirichlet eigenvalue problem of Laplacian for bounded domains in Euclidean space. For bounded domains in complete Riemannian manifolds, see the very nice book of Urakawa [28]. In this paper, we consider an eigenvalue problem of the biharmonic operator Δ^2 on a bounded domain with piecewise smooth boundary in an n -dimensional complete Riemannian manifold M , which is also called *the clamped plate problem*:

$$(1.2) \quad \begin{cases} \Delta^2 u = \Gamma u & \text{in } \Omega \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ^2 denotes the biharmonic operator on M , and ν is the outward unit normal of $\partial\Omega$.

When Ω is a bounded domain in \mathbf{R}^n , Agmon and Pleijel give the following asymptotic formula of eigenvalues of the clamped plate problem (1.2):

$$\Gamma_k \sim \frac{16\pi^4}{(\omega_n \text{vol}(\Omega))^{\frac{4}{n}}} k^{\frac{4}{n}}, \quad k \rightarrow \infty.$$

This implies that

$$\frac{1}{k} \sum_{j=1}^k \Gamma_j \sim \frac{n}{n+4} \frac{16\pi^4}{(\omega_n \text{vol}(\Omega))^{\frac{4}{n}}} k^{\frac{4}{n}}, \quad k \rightarrow \infty,$$

where Γ_j denotes the j^{th} eigenvalue of the clamped plate problem 1.2, $\text{vol}(\Omega)$ and ω_n denote volumes of Ω and the unit ball in \mathbf{R}^n , respectively. Furthermore, by making use of the Fourier transform and a lemma due to Hörmander, Levine and Protter [22] proves that eigenvalues of the clamped plate problem 1.2 satisfy

$$\frac{1}{k} \sum_{j=1}^k \Gamma_j \geq \frac{n}{n+4} \frac{16\pi^4}{(\omega_n \text{vol}(\Omega))^{\frac{4}{n}}} k^{\frac{4}{n}}.$$

The above formula shows that the coefficient of $k^{\frac{4}{n}}$ is the best possible constant. and the order of k is optimal according to the asymptotic formula of Agmon and Peijel. Cheng and Wei [12, 13] and Cheng, Qi and Wei [11] generalize the result of Levine and Protter by adding the lower terms.

On the other hand, it is a very difficult problem to obtain a sharp estimate for the upper bound of eigenvalues with optimal order of k of the clamped plate problem (1.2). For estimates for upper bounds of eigenvalues and estimates of two consecutive eigenvalues of the clamped plate problem, Payne, Pólya and Weinberger [24] proves

$$(1.3) \quad \Gamma_{k+1} - \Gamma_k \leq \frac{8(n+2)}{n^2 k} \sum_{i=1}^k \Gamma_i.$$

Chen and Qian [7] and Hook [21], independently, extend the above inequality to

$$(1.4) \quad \frac{n^2 k^2}{8(n+2)} \leq \sum_{i=1}^k \frac{\Gamma_i^{\frac{1}{2}}}{\Gamma_{k+1} - \Gamma_i} \sum_{i=1}^k \Gamma_i^{\frac{1}{2}}.$$

Cheng and Yang [15] and Wang and Xia [27] prove

$$(1.5) \quad \sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i)^2 \leq \frac{8(n+2)}{n^2} \sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i) \Gamma_i$$

In the open problem section (of the 6th International Chinese Congress of Mathematicians, July 9-14, 2013, Taiwan National University), the second author proposes the following problem:

Conjecture 1.1. Eigenvalues of the clamped plate problem (1.2) for a bounded domain in \mathbf{R}^n satisfies

$$(1.6) \quad \sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i)^2 \leq \frac{8}{n} \sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i) \Gamma_i.$$

In fact, if one may prove the conjecture 1.1, by making use of the recursive formula of Cheng and Yang [17], one may obtain the sharp estimates on the upper bound of the k^{th} eigenvalue, in the sense of the order of k , of the clamped plate problem.

In this paper, we study the gap of two consecutive eigenvalues of the clamped plate problem. We obtain the following:

Theorem 1.1. *Let Ω be a bounded domain in the Euclidean space \mathbf{R}^n . Then, for any integer $k \geq 0$, we have*

$$(1.7) \quad (\sqrt{\Gamma_{k+1} - \Gamma_1} - \sqrt{\Gamma_k - \Gamma_1})^2 \leq \frac{16\sqrt{\Gamma_1}}{n} \{(\Gamma_{k+1} - \Gamma_1)(\Gamma_k - \Gamma_1)\}^{\frac{1}{4}} + C,$$

where

$$C = \max \left\{ \frac{8 \int_{\Omega} |\nabla \Delta u_1|^2 dv}{(n+2) \|\nabla u_1\|^2}, \frac{4(n+12)\Gamma_1 + 16 \int_{\Omega} \sum_{m=1}^n \left(\frac{\partial^2 u_1}{\partial x_m^2}\right)^2 dv}{n} \right\}.$$

is constant only depending on the dimension n , the first eigenvalue Γ_1 and the normalized first eigenfunction u_1 .

Remark 1. *According to the asymptotic formula of Agmon and Pleijel, we have*

$$\lim_{k \rightarrow \infty} \frac{\Gamma_k}{k^{\frac{4}{n}}} = \frac{16\pi^4}{(\omega_n \text{vol}(\Omega))^{\frac{4}{n}}}.$$

From our theorem, we know, the gap between $\sqrt{\Gamma_{k+1} - \Gamma_1}$ and $\sqrt{\Gamma_k - \Gamma_1}$ is bounded by a term with a lower order $k^{\frac{1}{n}}$ in the sense of the asymptotic formula of Agmon and Peijel.

Since

$$\Gamma_{k+1} - \Gamma_k = (\sqrt{\Gamma_{k+1} - \Gamma_1} - \sqrt{\Gamma_k - \Gamma_1})(\sqrt{\Gamma_{k+1} - \Gamma_1} + \sqrt{\Gamma_k - \Gamma_1}),$$

according to the asymptotic formula of Agmon and Pleijel, we know that the gap between Γ_{k+1} and Γ_k is bounded by a term with a lower order $k^{\frac{3}{n}}$.

2. A GENERAL RESULT

Let Ω be a bounded domain with piecewise smooth boundary in an n -dimensional complete Riemannian manifold M . Let u_i be an eigenfunction corresponding to the eigenvalue Γ_i such that

$$(2.1) \quad \begin{cases} \Delta^2 u_i = \Gamma_i u_i & \text{in } \Omega \\ u_i = \frac{\partial u_i}{\partial \nu} = 0 & \text{on } \partial\Omega \\ \int_{\Omega} u_i u_j dv = \delta_{ij}, \quad i, j = 1, 2, \dots, \end{cases}$$

where eigenvalues are accounted according to their multiplicities. Thus, we know that $\{u_j\}_{j=1}^{\infty}$ forms an orthonormal base of $L^2(\Omega)$ -function space. For any smooth function g , we can write

$$gu_1 = \sum_{j=1}^{\infty} r_j u_j, \quad \|gu_1\|^2 = \int_{\Omega} (gu_1)^2 dv = \sum_{j=1}^{\infty} r_j^2,$$

where $r_j = \int_{\Omega} gu_1 u_j dv$, for $j = 1, 2, \dots$. For any positive integer k , we define

$$(2.2) \quad \varphi := gu_1 - \sum_{j=1}^k r_j u_j.$$

By a simple calculation, we obtain

$$(2.3) \quad \int_{\Omega} u_j \varphi dv = 0, \quad j = 1, \dots, k.$$

Hence

$$\|\varphi\|^2 = \sum_{j=k+1}^{\infty} r_j^2.$$

Defining

$$(2.4) \quad p = \Delta^2 g \cdot u_1 + 2\nabla(\Delta g) \cdot \nabla u_1 + 2\Delta g \Delta u_1 + 2\Delta(\nabla g \cdot \nabla u_1) + 2\nabla g \cdot \nabla(\Delta u_1),$$

we have

$$p = \sum_{j=1}^{\infty} s_j u_j, \quad \|p\|^2 = \sum_{j=1}^{\infty} s_j^2,$$

where

$$s_j = \int_{\Omega} p u_j dv.$$

Since

$$\begin{aligned} & 2 \int_{\Omega} (\Delta u_j \nabla g \cdot \nabla u_1 - \Delta u_1 \nabla g \cdot \nabla u_j) dv \\ &= (\Gamma_j - \Gamma_1) r_j - \int_{\Omega} u_1 \Delta u_j \Delta g dv + \int_{\Omega} u_j \Delta u_1 \Delta g dv, \end{aligned}$$

we can infer

$$(2.5) \quad s_j = (\Gamma_j - \Gamma_1)r_j.$$

Thus, we get

$$\begin{aligned} \|p\|^2 &= \sum_{j=1}^{\infty} (\Gamma_j - \Gamma_1)^2 r_j^2. \\ \int_{\Omega} gu_1 p dv &= \int_{\Omega} gu_1 \sum_{j=1}^{\infty} s_j u_j dv = \sum_{j=1}^{\infty} s_j r_j = \sum_{j=1}^{\infty} (\Gamma_j - \Gamma_1) r_j^2. \end{aligned}$$

From the definition of φ , we have

$$\int_{\Omega} \varphi p dv = \int_{\Omega} (gu_1 - \sum_{j=1}^k r_j u_j) p dv = \sum_{j=1}^{\infty} (\Gamma_j - \Gamma_1) r_j^2 - \sum_{j=1}^k (\Gamma_j - \Gamma_1) r_j^2.$$

Hence, we obtain

$$\int_{\Omega} \varphi p dv = \sum_{j=k+1}^{\infty} (\Gamma_j - \Gamma_1) r_j^2.$$

The following algebraic lemma plays an important role in this paper, which may be found in Chen-Yang-Zheng [6], essentially. For reader's convenient, we give a detailed proof of it in the Appendix.

Lemma 2.1. *Let $\{\mu_j\}_{j=k+1}^{\infty}$ be a sequence satisfying*

$$0 \leq \mu_{k+1} \leq \mu_{k+2} \leq \cdots \rightarrow \infty.$$

If a sequence $\{a_j\}_{j=k+1}^{\infty}$ satisfies $\sum_{j=k+1}^{\infty} \mu_j^2 a_j^2 = A < \infty$ and $\sum_{j=k+1}^{\infty} a_j^2 = B < \infty$, then we have

$$\sum_{j=k+1}^{\infty} \mu_j a_j^2 \leq \frac{A + \mu_{k+1} \mu_{k+2} B}{\mu_{k+1} + \mu_{k+2}}.$$

By applying the lemma 2.1 with $\mu_j = \Gamma_j - \Gamma_1$ and $a_j = r_j$, we obtain

$$\begin{aligned} &\{(\Gamma_{k+1} - \Gamma_1) + (\Gamma_{k+2} - \Gamma_1)\} \int_{\Omega} \varphi p dv \\ &\leq (\|p\|^2 - \sum_{j=1}^k (\Gamma_j - \Gamma_1)^2 r_j^2) + (\Gamma_{k+1} - \Gamma_1)(\Gamma_{k+2} - \Gamma_1) \|\varphi\|^2, \end{aligned}$$

namely,

$$\begin{aligned} &\{(\Gamma_{k+1} - \Gamma_1) + (\Gamma_{k+2} - \Gamma_1)\} \left(\int_{\Omega} gu_1 p dv - \sum_{j=1}^k (\Gamma_j - \Gamma_1) r_j^2 \right) \\ &\leq (\|p\|^2 - \sum_{j=1}^k (\Gamma_j - \Gamma_1)^2 r_j^2) + (\Gamma_{k+1} - \Gamma_1)(\Gamma_{k+2} - \Gamma_1) (\|gu_1\|^2 - \sum_{j=1}^k r_j^2). \end{aligned}$$

Since

$$\begin{aligned} & \{(\Gamma_{k+1} - \Gamma_1) + (\Gamma_{k+2} - \Gamma_1)\} \sum_{j=1}^k (\Gamma_j - \Gamma_1) r_j^2 \\ & \leq \sum_{j=1}^k (\Gamma_j - \Gamma_1)^2 r_j^2 + (\Gamma_{k+1} - \Gamma_1)(\Gamma_{k+2} - \Gamma_1) \sum_{j=1}^k r_j^2, \end{aligned}$$

we have

$$\{(\Gamma_{k+1} - \Gamma_1) + (\Gamma_{k+2} - \Gamma_1)\} \int_{\Omega} g u_1 p dv \leq \|p\|^2 + (\Gamma_{k+1} - \Gamma_1)(\Gamma_{k+2} - \Gamma_1) \|g u_1\|^2.$$

Thus, we have proved the following:

Theorem 2.1. *Let Ω be a bounded domain in an n -dimensional complete Riemannian manifold M . Assume that Γ_i is the i^{th} eigenvalue of the clamped plate problem (1.2). For any smooth function g , we have, for any integer k ,*

$$\{(\Gamma_{k+2} - \Gamma_1) + (\Gamma_{k+1} - \Gamma_1)\} \int_{\Omega} g u_1 p dv \leq \|p\|^2 + (\Gamma_{k+2} - \Gamma_1)(\Gamma_{k+1} - \Gamma_1) \|g u_1\|^2,$$

where p is defined by the formula (2.4) and u_1 is the normalized first eigenfunction corresponding to the first eigenvalue Γ_1 .

Lemma 2.2.

$$\int_{\Omega} g u_1 p dv = \int_{\Omega} \left\{ (\Delta g)^2 u_1^2 + 4(\nabla g \cdot \nabla u_1)^2 - 2|\nabla g|^2 u_1 \Delta u_1 + 4u_1 \Delta g \nabla g \cdot \nabla u_1 \right\} dv.$$

Proof. From Stokes' theorem, we infer

$$\begin{aligned} 2 \int_{\Omega} g u_1 \nabla(\Delta g) \cdot \nabla u_1 dv &= \int_{\Omega} \left\{ 2u_1 \Delta g \nabla u_1 \cdot \nabla g + u_1^2 (\Delta g)^2 - g u_1^2 \Delta^2 g \right\} dv, \\ 2 \int_{\Omega} g u_1 \Delta(\nabla g \cdot \nabla u_1) dv &= \int_{\Omega} \left\{ 2u_1 \Delta g \nabla g \cdot \nabla u_1 + 4(\nabla g \cdot \nabla u_1)^2 + 2g \Delta u_1 \nabla g \cdot \nabla u_1 \right\} dv, \\ 2 \int_{\Omega} g u_1 \nabla g \cdot \nabla(\Delta u_1) dv &= -2 \int_{\Omega} \left(|\nabla g|^2 u_1 \Delta u_1 + g \Delta u_1 \nabla g \cdot \nabla u_1 + g \Delta g u_1 \Delta u_1 \right) dv. \end{aligned}$$

From the definition of p , we obtain

$$\int_{\Omega} g u_1 p dv = \int_{\Omega} \left\{ (\Delta g)^2 u_1^2 + 4(\nabla g \cdot \nabla u_1)^2 - 2|\nabla g|^2 u_1 \Delta u_1 + 4u_1 \Delta g \nabla g \cdot \nabla u_1 \right\} dv.$$

□

For any smooth function f in M and constant a , we consider $g_1 = \cos(af)$. We have

$$\nabla g_1 = -a \sin(af) \nabla f, \quad \Delta g_1 = -a^2 \cos(af) |\nabla f|^2 - a \sin(af) \Delta f$$

$$\begin{aligned} \nabla \Delta g_1 &= a^3 \sin(af) |\nabla f|^2 \nabla f - a^2 \cos(af) \nabla(|\nabla f|^2) \\ &\quad - a^2 \cos(af) \Delta f \nabla f - a \sin(af) \nabla(\Delta f) \end{aligned}$$

$$\begin{aligned}
\Delta^2 g_1 &= a^4 \cos(af) |\nabla f|^4 + 2a^3 \sin(af) \nabla(|\nabla f|^2) \cdot \nabla f + 2a^3 \sin(af) |\nabla f|^2 \Delta f \\
&\quad - a^2 \cos(af) \Delta(|\nabla f|^2) - 2a^2 \cos(af) \nabla(\Delta f) \cdot \nabla f \\
&\quad - a^2 \cos(af) (\Delta f)^2 - a \sin(af) \Delta^2 f.
\end{aligned}$$

In the same way, for $g_2 = \sin(af)$, we have

$$\nabla g_2 = a \cos(af) \nabla f, \quad \Delta g_2 = -a^2 \sin(af) |\nabla f|^2 + a \cos(af) \Delta f$$

$$\begin{aligned}
\nabla \Delta g_2 &= -a^3 \cos(af) |\nabla f|^2 \nabla f - a^2 \sin(af) \nabla(|\nabla f|^2) \\
&\quad - a^2 \sin(af) \Delta f \nabla f + a \cos(af) \nabla(\Delta f)
\end{aligned}$$

$$\begin{aligned}
\Delta^2 g_2 &= a^4 \sin(af) |\nabla f|^4 - 2a^3 \cos(af) \nabla(|\nabla f|^2) \cdot \nabla f \\
&\quad - 2a^3 \cos(af) |\nabla f|^2 \Delta f - a^2 \sin(af) \Delta(|\nabla f|^2) \\
&\quad - 2a^2 \sin(af) \nabla(\Delta f) \cdot \nabla f - a^2 \sin(af) (\Delta f)^2 + a \cos(af) \Delta^2 f.
\end{aligned}$$

Thus, we obtain the following:

Lemma 2.3. *If the function f satisfies $|\nabla f|^2 = 1$ and $\Delta f = b = \text{constant}$, we have*

$$\nabla g_1 = -a \sin(af) \nabla f, \quad \Delta g_1 = -a^2 \cos(af) - ab \sin(af),$$

$$\nabla \Delta g_1 = a^3 \sin(af) \nabla f - a^2 b \cos(af) \nabla f,$$

$$\Delta^2 g_1 = a^4 \cos(af) + 2a^3 b \sin(af) - a^2 b^2 \cos(af),$$

$$\nabla g_2 = a \cos(af) \nabla f, \quad \Delta g_2 = -a^2 \sin(af) + ab \cos(af),$$

$$\nabla \Delta g_2 = -a^3 \cos(af) \nabla f - a^2 b \sin(af) \nabla f,$$

$$\Delta^2 g_2 = a^4 \sin(af) - 2a^3 b \cos(af) - a^2 b^2 \sin(af).$$

By defining

$$(2.6) \quad p_1 = \Delta^2 g_1 \cdot u_1 + 2\nabla(\Delta g_1) \cdot \nabla u_1 + 2\Delta g_1 \Delta u_1 + 2\Delta(\nabla g_1 \cdot \nabla u_1) + 2\nabla g_1 \cdot \nabla(\Delta u_1),$$

and

$$(2.7) \quad p_2 = \Delta^2 g_2 \cdot u_1 + 2\nabla(\Delta g_2) \cdot \nabla u_1 + 2\Delta g_2 \Delta u_1 + 2\Delta(\nabla g_2 \cdot \nabla u_1) + 2\nabla g_2 \cdot \nabla(\Delta u_1),$$

we have

Proposition 2.1. *If the function f satisfies $|\nabla f|^2 = 1$ and $\Delta f = b = \text{constant}$, we have*

$$\begin{aligned}
&|p_1|^2 + |p_2|^2 \\
&= \left((a^4 - a^2 b^2) u_1 - 4a^2 b \nabla f \cdot \nabla u_1 - 2a^2 \Delta u_1 - 4a^2 \nabla f \cdot \nabla(\nabla f \cdot \nabla u_1) \right)^2 \\
&\quad + \left(2a^3 b u_1 + 4a^3 \nabla f \cdot \nabla u_1 - 2ab \Delta u_1 - 2a \Delta(\nabla f \cdot \nabla u_1) - 2a \nabla f \cdot \nabla(\Delta u_1) \right)^2.
\end{aligned}$$

Proof. From the above lemma 2.3, we have

$$\begin{aligned} p_1 &= (a^4 \cos(af) + 2a^3 b \sin(af) - a^2 b^2 \cos(af))u_1 \\ &\quad + 2(a^3 \sin(af) - a^2 b \cos(af))\nabla f \cdot \nabla u_1 - 2(a^2 \cos(af) + ab \sin(af))\Delta u_1 \\ &\quad - 2a\Delta(\sin(af)\nabla f \cdot \nabla u_1) - 2a \sin(af)\nabla f \cdot \nabla(\Delta u_1) \end{aligned}$$

and

$$\begin{aligned} \Delta(\sin(af)\nabla f \cdot \nabla u_1) &= \sin(af)\Delta(\nabla f \cdot \nabla u_1) \\ &\quad + 2a \cos(af)\nabla f \cdot \nabla(\nabla f \cdot \nabla u_1) - \left(a^2 \sin(af) - ab \cos(af)\right)\nabla f \cdot \nabla u_1, \end{aligned}$$

$$\begin{aligned} p_2 &= (a^4 \sin(af) - 2a^3 b \cos(af) - a^2 b^2 \sin(af))u_1 \\ &\quad - 2(a^3 \cos(af) + a^2 b \sin(af))\nabla f \cdot \nabla u_1 - 2(a^2 \sin(af) - ab \cos(af))\Delta u_1 \\ &\quad + 2a\Delta(\cos(af)\nabla f \cdot \nabla u_1) + 2a \cos(af)\nabla f \cdot \nabla(\Delta u_1) \end{aligned}$$

and

$$\begin{aligned} \Delta(\cos(af)\nabla f \cdot \nabla u_1) &= \cos(af)\Delta(\nabla f \cdot \nabla u_1) \\ &\quad - 2a \sin(af)\nabla f \cdot \nabla(\nabla f \cdot \nabla u_1) - \left(a^2 \cos(af) + ab \sin(af)\right)\nabla f \cdot \nabla u_1. \end{aligned}$$

Hence, we infer

$$\begin{aligned} p_1 &= \left((a^4 - a^2 b^2)u_1 - 4a^2 b \nabla f \cdot \nabla u_1 - 2a^2 \Delta u_1 - 4a^2 \nabla f \cdot \nabla(\nabla f \cdot \nabla u_1)\right)\cos(af) \\ &\quad + \left(2a^3 b u_1 + 4a^3 \nabla f \cdot \nabla u_1 - 2ab \Delta u_1 - 2a\Delta(\nabla f \cdot \nabla u_1) - 2a \nabla f \cdot \nabla(\Delta u_1)\right)\sin(af), \\ p_2 &= \left((a^4 - a^2 b^2)u_1 - 4a^2 b \nabla f \cdot \nabla u_1 - 2a^2 \Delta u_1 - 4a^2 \nabla f \cdot \nabla(\nabla f \cdot \nabla u_1)\right)\sin(af) \\ &\quad - \left(2a^3 b u_1 + 4a^3 \nabla f \cdot \nabla u_1 - 2ab \Delta u_1 - 2a\Delta(\nabla f \cdot \nabla u_1) - 2a \nabla f \cdot \nabla(\Delta u_1)\right)\cos(af). \end{aligned}$$

From the above two equalities, we obtain

$$\begin{aligned} &|p_1|^2 + |p_2|^2 \\ &= \left((a^4 - a^2 b^2)u_1 - 4a^2 b \nabla f \cdot \nabla u_1 - 2a^2 \Delta u_1 - 4a^2 \nabla f \cdot \nabla(\nabla f \cdot \nabla u_1)\right)^2 \\ &\quad + \left(2a^3 b u_1 + 4a^3 \nabla f \cdot \nabla u_1 - 2ab \Delta u_1 - 2a\Delta(\nabla f \cdot \nabla u_1) - 2a \nabla f \cdot \nabla(\Delta u_1)\right)^2. \end{aligned}$$

□

Proposition 2.2. *If the function f satisfies $|\nabla f|^2 = 1$ and $\Delta f = b = \text{constant}$, we have*

$$\int_{\Omega} g_1 u_1 p_1 dv + \int_{\Omega} g_2 u_1 p_2 dv = \int_{\Omega} \left\{ (a^4 - a^2 b^2)u_1^2 + 4a^2 (\nabla f \cdot \nabla u_1)^2 - 2a^2 u_1 \Delta u_1 \right\} dv.$$

Proof. Since

$$\begin{aligned} \int_{\Omega} g_1 u_1 p_1 dv &= \int_{\Omega} \left\{ (a^2 \cos(af) + ab \sin(af))^2 u_1^2 \right. \\ &\quad + 4a^2 (\sin(af))^2 (\nabla f \cdot \nabla u_1)^2 - 2a^2 (\sin(af))^2 u_1 \Delta u_1 \\ &\quad \left. + 4a \sin(af) (a^2 \cos(af) + ab \sin(af)) u_1 \nabla f \cdot \nabla u_1 \right\} dv \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega} g_2 u_1 p_2 dv &= \int_{\Omega} \left\{ (a^2 \sin(af) - ab \cos(af))^2 u_1^2 \right. \\ &\quad + 4a^2 (\cos(af))^2 (\nabla f \cdot \nabla u_1)^2 - 2a^2 (\cos(af))^2 u_1 \Delta u_1 \\ &\quad \left. - 4a \cos(af) (a^2 \sin(af) - ab \cos(af)) u_1 \nabla f \cdot \nabla u_1 \right\} dv, \end{aligned}$$

we infer

$$\begin{aligned} &\int_{\Omega} g_1 u_1 p_1 dv + \int_{\Omega} g_2 u_1 p_2 dv \\ &= \int_{\Omega} \left\{ (a^4 + a^2 b^2) u_1^2 + 4a^2 (\nabla f \cdot \nabla u_1)^2 - 2a^2 u_1 \Delta u_1 + 4a^2 b u_1 \nabla f \cdot \nabla u_1 \right\} dv. \end{aligned}$$

According to Stokes formula, we know

$$\int_{\Omega} 2u_1 \nabla f \cdot \nabla u_1 dv = - \int_{\Omega} b u_1^2 dv.$$

Hence, we get

$$\int_{\Omega} g_1 u_1 p_1 dv + \int_{\Omega} g_2 u_1 p_2 dv = \int_{\Omega} \left\{ (a^4 - a^2 b^2) u_1^2 + 4a^2 (\nabla f \cdot \nabla u_1)^2 - 2a^2 u_1 \Delta u_1 \right\} dv.$$

□

3. THE PROOF OF THE THEOREM 1.1

Proof of Theorem 1.1. Since Ω is a bounded domain in the Euclidean space \mathbf{R}^n . Let (x_1, x_2, \dots, x_n) be the standard coordinate. By taking $f = x_m$, for $m = 1, 2, \dots, n$, we know

$$|\nabla f|^2 = 1, \quad \Delta f = 0.$$

Thus, from the propositions 2.1, we obtain, for $m = 1, 2, \dots, n$,

$$\begin{aligned} |p_1|^2 + |p_2|^2 &= \left(a^4 u_1 - 2a^2 \Delta u_1 - 4a^2 \frac{\partial^2 u_1}{\partial x_m^2} \right)^2 \\ &\quad + \left(4a^3 \frac{\partial u_1}{\partial x_m} - 2a \Delta \left(\frac{\partial u_1}{\partial x_m} \right) - 2a \frac{\partial (\Delta u_1)}{\partial x_m} \right)^2 \\ &= a^4 \left(a^2 u_1 - 2 \Delta u_1 - 4 \frac{\partial^2 u_1}{\partial x_m^2} \right)^2 + 16a^2 \left(a^2 \frac{\partial u_1}{\partial x_m} - \Delta \left(\frac{\partial u_1}{\partial x_m} \right) \right)^2. \end{aligned}$$

Hence,

$$\begin{aligned} \int_{\Omega} |p_1|^2 dv + \int_{\Omega} |p_2|^2 dv &= \int_{\Omega} a^4 \left(a^2 u_1 - 2\Delta u_1 - 4 \frac{\partial^2 u_1}{\partial x_m^2} \right)^2 dv \\ &\quad + \int_{\Omega} 16a^2 \left(a^2 \frac{\partial u_1}{\partial x_m} - \Delta \left(\frac{\partial u_1}{\partial x_m} \right) \right)^2 dv \end{aligned}$$

holds. By a direct computation, we infer

$$\begin{aligned} &\int_{\Omega} \left(a^2 u_1 - 2\Delta u_1 - 4 \frac{\partial^2 u_1}{\partial x_m^2} \right)^2 dv \\ &= a^4 + 4\Gamma_1 + 16 \int_{\Omega} \left(\frac{\partial^2 u_1}{\partial x_m^2} \right)^2 dv \\ &\quad + 4a^2 \int_{\Omega} |\nabla u_1|^2 dv - 8a^2 \int_{\Omega} u_1 \frac{\partial^2 u_1}{\partial x_m^2} dv + 16 \int_{\Omega} \Delta u_1 \frac{\partial^2 u_1}{\partial x_m^2} dv \end{aligned}$$

and

$$\begin{aligned} &\int_{\Omega} \left(a^2 \frac{\partial u_1}{\partial x_m} - \Delta \left(\frac{\partial u_1}{\partial x_m} \right) \right)^2 dv \\ &= \int_{\Omega} \left(a^4 \left(\frac{\partial u_1}{\partial x_m} \right)^2 + \left(\Delta \left(\frac{\partial u_1}{\partial x_m} \right) \right)^2 - 2a^2 \frac{\partial u_1}{\partial x_m} \Delta \left(\frac{\partial u_1}{\partial x_m} \right) \right) dv. \end{aligned}$$

We derive

$$\begin{aligned} &\int_{\Omega} |p_1|^2 dv + \int_{\Omega} |p_2|^2 dv \\ &= a^4 \left\{ a^4 + 4\Gamma_1 + 16 \int_{\Omega} \left(\frac{\partial^2 u_1}{\partial x_m^2} \right)^2 dv + 4a^2 \int_{\Omega} |\nabla u_1|^2 dv \right. \\ &\quad \left. - 8a^2 \int_{\Omega} u_1 \frac{\partial^2 u_1}{\partial x_m^2} dv + 16 \int_{\Omega} \Delta u_1 \frac{\partial^2 u_1}{\partial x_m^2} dv \right\} \\ &\quad + 16a^2 \left\{ \int_{\Omega} \left(a^4 \left(\frac{\partial u_1}{\partial x_m} \right)^2 + \left(\Delta \left(\frac{\partial u_1}{\partial x_m} \right) \right)^2 - 2a^2 \frac{\partial u_1}{\partial x_m} \Delta \left(\frac{\partial u_1}{\partial x_m} \right) \right) dv \right\}. \end{aligned}$$

From the proposition 2.2, we infer

$$\begin{aligned} \int_{\Omega} g_1 u_1 p_1 dv + \int_{\Omega} g_2 u_1 p_2 dv &= \int_{\Omega} \left\{ a^4 u_1^2 + 4a^2 \left(\frac{\partial u_1}{\partial x_m} \right)^2 - 2a^2 u_1 \Delta u_1 \right\} dv \\ &= a^4 + 2a^2 \int_{\Omega} \left\{ 2 \left(\frac{\partial u_1}{\partial x_m} \right)^2 + |\nabla u_1|^2 \right\} dv. \end{aligned}$$

We apply the theorem 2.1 to functions $g = g_1$ and $g = g_2$, respectively and take summation for them, we have

$$\begin{aligned}
& \{(\Gamma_{k+1} - \Gamma_1) + (\Gamma_{k+2} - \Gamma_1)\} \left(a^4 + 2a^2 \int_{\Omega} \left\{ 2\left(\frac{\partial u_1}{\partial x_m}\right)^2 + |\nabla u_1|^2 \right\} dv \right) \\
& \leq a^4 \left\{ a^4 + 4\Gamma_1 + 16 \int_{\Omega} \left(\frac{\partial^2 u_1}{\partial x_m^2}\right)^2 dv + 4a^2 \int_{\Omega} |\nabla u_1|^2 dv \right. \\
(3.1) \quad & \left. - 8a^2 \int_{\Omega} u_1 \frac{\partial^2 u_1}{\partial x_m^2} dv + 16 \int_{\Omega} \Delta u_1 \frac{\partial^2 u_1}{\partial x_m^2} dv \right\} \\
& + 16a^2 \left\{ \int_{\Omega} \left(a^4 \left(\frac{\partial u_1}{\partial x_m}\right)^2 + \left(\Delta \left(\frac{\partial u_1}{\partial x_m}\right)\right)^2 - 2a^2 \frac{\partial u_1}{\partial x_m} \Delta \left(\frac{\partial u_1}{\partial x_m}\right) \right) dv \right\} \\
& + (\Gamma_{k+1} - \Gamma_1)(\Gamma_{k+2} - \Gamma_1),
\end{aligned}$$

Taking summation for m from 1 to n and making use of Stokes formula, we have

$$\begin{aligned}
& \{(\Gamma_{k+1} - \Gamma_1) + (\Gamma_{k+2} - \Gamma_1)\} (na^4 + 2a^2(2+n)\|\nabla u_1\|^2) \\
& \leq a^4 \left\{ na^4 + 4(n+4)\Gamma_1 + 4a^2(n+2)\|\nabla u_1\|^2 + 16 \int_{\Omega} \sum_{m=1}^n \left(\frac{\partial^2 u_1}{\partial x_m^2}\right)^2 dv \right\} \\
(3.2) \quad & + 16a^2 \left\{ a^4 \|\nabla u_1\|^2 + 2a^2 \Gamma_1 + \int_{\Omega} \sum_{m=1}^n \left(\Delta \left(\frac{\partial u_1}{\partial x_m}\right)\right)^2 dv \right\} \\
& + n(\Gamma_{k+1} - \Gamma_1)(\Gamma_{k+2} - \Gamma_1),
\end{aligned}$$

that is,

$$\begin{aligned}
& \{(\Gamma_{k+1} - \Gamma_1) + (\Gamma_{k+2} - \Gamma_1)\} (na^2 + 2(n+2)\|\nabla u_1\|^2) \\
& \leq a^2 \left\{ na^4 + 4a^2(n+6)\|\nabla u_1\|^2 + 4(n+12)\Gamma_1 + 16 \int_{\Omega} \sum_{m=1}^n \left(\frac{\partial^2 u_1}{\partial x_m^2}\right)^2 dv \right\} \\
& + 16 \int_{\Omega} \sum_{m=1}^n \left(\Delta \left(\frac{\partial u_1}{\partial x_m}\right)\right)^2 dv + \frac{n}{a^2} (\Gamma_{k+1} - \Gamma_1)(\Gamma_{k+2} - \Gamma_1).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
& \{(\Gamma_{k+1} - \Gamma_1) + (\Gamma_{k+2} - \Gamma_1)\} \\
& \leq a^2 \left(a^2 + 2\frac{n+2}{n} \|\nabla u_1\|^2 \right) + \frac{(\Gamma_{k+1} - \Gamma_1)(\Gamma_{k+2} - \Gamma_1)}{a^2(a^2 + 2\frac{n+2}{n} \|\nabla u_1\|^2)} \\
& + \frac{16a^4 \|\nabla u_1\|^2}{n(a^2 + 2\frac{n+2}{n} \|\nabla u_1\|^2)} \\
& + \frac{16 \int_{\Omega} |\nabla \Delta u_1|^2 + \left(4(n+12)\Gamma_1 + 16 \int_{\Omega} \sum_{m=1}^n \left(\frac{\partial^2 u_1}{\partial x_m^2}\right)^2 dv \right) a^2}{na^2 + 2(n+2)\|\nabla u_1\|^2}.
\end{aligned}$$

For $k_1 \geq 0$, $k_2 > 0$ and $k_3 > 0$, the function $f(t) = \frac{k_1 + tk_2}{nt + k_3}$, for $t \geq 0$, satisfies

$$f(t) \leq \max\left\{\frac{k_1}{k_3}, \frac{k_2}{n}\right\}.$$

Thus, we have

$$\frac{16 \int_{\Omega} |\nabla \Delta u_1|^2 + \left(4(n+12)\Gamma_1 + 16 \int_{\Omega} \sum_{m=1}^n \left(\frac{\partial^2 u_1}{\partial x_m^2} \right)^2 dv \right) a^2}{na^2 + 2(n+2)\|\nabla u_1\|^2} \leq C,$$

where C is given by

$$C = \max \left\{ \frac{8 \int_{\Omega} |\nabla \Delta u_1|^2 dv}{(n+2)\|\nabla u_1\|^2}, \frac{4(n+12)\Gamma_1 + 16 \int_{\Omega} \sum_{m=1}^n \left(\frac{\partial^2 u_1}{\partial x_m^2} \right)^2 dv}{n} \right\}.$$

If we put

$$a^2 \left(a^2 + 2 \frac{n+2}{n} \|\nabla u_1\|^2 \right) = \sqrt{(\Gamma_{k+1} - \Gamma_1)(\Gamma_{k+2} - \Gamma_1)},$$

we obtain

$$a^4 \leq \sqrt{(\Gamma_{k+1} - \Gamma_1)(\Gamma_{k+2} - \Gamma_1)},$$

$$\left(\sqrt{\Gamma_{k+2} - \Gamma_1} - \sqrt{\Gamma_{k+1} - \Gamma_1} \right)^2 \leq \frac{16\sqrt{\Gamma_1}}{n} \left\{ (\Gamma_{k+1} - \Gamma_1)(\Gamma_{k+2} - \Gamma_1) \right\}^{\frac{1}{4}} + C,$$

because

$$\|\nabla u_1\|^2 \leq \sqrt{\Gamma_1}.$$

If we change $k+2$ and $k+1$ into $k+1$ and k , respectively, we know that the theorem 1.1 is proved.

4. APPENDIX

In this Appendix, we shall give a proof of the lemma 2.1.

Lemma 2.1. Let $\{\mu_j\}_{j=k+1}^{\infty}$ be a sequence satisfying

$$0 \leq \mu_{k+1} \leq \mu_{k+2} \leq \cdots \rightarrow \infty.$$

If a sequence $\{a_j\}_{j=k+1}^{\infty}$ satisfies $\sum_{j=k+1}^{\infty} \mu_j^2 a_j^2 = A < \infty$ and $\sum_{j=k+1}^{\infty} a_j^2 = B < \infty$, then we have

$$\sum_{j=k+1}^{\infty} \mu_j a_j^2 \leq \frac{A + \mu_{k+1} \mu_{k+2} B}{\mu_{k+1} + \mu_{k+2}}.$$

Proof. From the Cauchy-Schwarz inequality, we know

$$\mu_{k+1} \sum_{j=k+1}^{\infty} a_j^2 \leq \sum_{j=k+1}^{\infty} \mu_j a_j^2 \leq \sqrt{\sum_{j=k+1}^{\infty} \mu_j^2 a_j^2 \sum_{j=k+1}^{\infty} a_j^2} = \sqrt{AB}.$$

Hence

$$\mu_{k+1} \leq \sqrt{\frac{A}{B}}.$$

For any sequence $\{x_j\}_{j=k+1}^{\infty}$ with $\sum_{j=k+1}^{\infty} \mu_j^2 x_j^2 = A$ and $\sum_{j=k+1}^{\infty} x_j^2 = B$, we consider the following function

$$F(x_j) = \sum_{j=k+1}^{\infty} \mu_j x_j^2 + \lambda \left(\sum_{j=k+1}^{\infty} \mu_j^2 x_j^2 - A \right) + \mu \left(\sum_{j=k+1}^{\infty} x_j^2 - B \right),$$

where λ and μ are Lagrange multipliers. Thus, the maximum f_{max} of the function $f = \sum_{j=k+1}^{\infty} \mu_j x_j^2$ is attained at critical points of F . If $\{c_j\}_{j=k+1}^{\infty}$ is a critical point of F , for any sequence $\{b_j\}_{j=k+1}^{\infty}$, we have

$$\left. \frac{dF(c_j + tb_j)}{dt} \right|_{t=0} = 2 \sum_{j=k+1}^{\infty} \mu_j c_j b_j + 2\lambda \sum_{j=k+1}^{\infty} \mu_j^2 c_j b_j + 2\mu \sum_{j=k+1}^{\infty} c_j b_j = 0.$$

By taking

$$b_j = \begin{cases} 1 & j = p, \\ 0 & j \neq p, \end{cases}$$

we have

$$(\mu_p + \lambda\mu_p^2 + \mu)c_p = 0.$$

Since $\mu_p + \lambda\mu_p^2 + \mu = 0$ is a quadratic equation of μ_p , if $\mu_p + \lambda\mu_p^2 + \mu \neq 0$, we have $c_p = 0$. Let μ_r and μ_s , $r < s$, be solutions of $\mu_p + \lambda\mu_p^2 + \mu = 0$ with multiplicity $r_0 + 1$ and $s_0 + 1$, respectively, that is,

$$\mu_r = \mu_{r+1} = \cdots = \mu_{r+r_0} \quad \mu_s = \mu_{s+1} = \cdots = \mu_{s+s_0}.$$

Therefore, we have

$$\begin{aligned} A &= \mu_r^2(c_r^2 + c_{r+1}^2 + \cdots + c_{r+r_0}^2) + \mu_s^2(c_s^2 + c_{s+1}^2 + \cdots + c_{s+s_0}^2), \\ (4.1) \quad B &= (c_r^2 + c_{r+1}^2 + \cdots + c_{r+r_0}^2) + (c_s^2 + c_{s+1}^2 + \cdots + c_{s+s_0}^2), \\ f_{max} &= \mu_r(c_r^2 + c_{r+1}^2 + \cdots + c_{r+r_0}^2) + \mu_s(c_s^2 + c_{s+1}^2 + \cdots + c_{s+s_0}^2) \end{aligned}$$

Hence, we get

$$f_{max} = \frac{A + \mu_r \mu_s B}{\mu_r + \mu_s}.$$

Since $f_{max} \leq \sqrt{AB}$ from the Cauchy-Schwarz inequality, we have

$$f_{max} = \frac{A + \mu_r \mu_s B}{\mu_r + \mu_s} \leq \sqrt{AB}.$$

Thus, we obtain

$$(\sqrt{\frac{A}{B}} - \mu_r)(\sqrt{\frac{A}{B}} - \mu_s) \leq 0,$$

that is, we have

$$\sqrt{\frac{A}{B}} - \mu_r \geq 0, \quad \sqrt{\frac{A}{B}} - \mu_s \leq 0$$

because of $\mu_r \leq \mu_s$. Since $\sqrt{\frac{A}{B}} - \mu_r \geq 0$, we know that $G(t) = \frac{A + \mu_r t B}{\mu_r + t}$ is a decreasing function of t . Hence, we have

$$f_{max} \leq \frac{A + \mu_r \mu_{k+2} B}{\mu_r + \mu_{k+2}}.$$

If $\mu_{k+2} \geq \sqrt{\frac{A}{B}}$, we have $\mu_r = \mu_{k+1}$ because of $r < s$ and $\mu_r \leq \sqrt{\frac{A}{B}}$, that is

$$f_{max} \leq \frac{A + \mu_{k+1} \mu_{k+2} B}{\mu_{k+1} + \mu_{k+2}}.$$

If $\mu_{k+2} \leq \sqrt{\frac{A}{B}}$, we know that $G(t) = \frac{A + \mu_{k+2}tB}{\mu_{k+2} + t}$ is a decreasing function of t . Hence, we have

$$f_{max} \leq \frac{A + \mu_r \mu_{k+2} B}{\mu_r + \mu_{k+2}} \leq \frac{A + \mu_{k+1} \mu_{k+2} B}{\mu_{k+1} + \mu_{k+2}}.$$

It completes the proof of the lemma. □

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